Root Locus Design Method

Part A: Introduction

Goal

Learn a *specific technique* which shows how

changes in one of a system's parameter

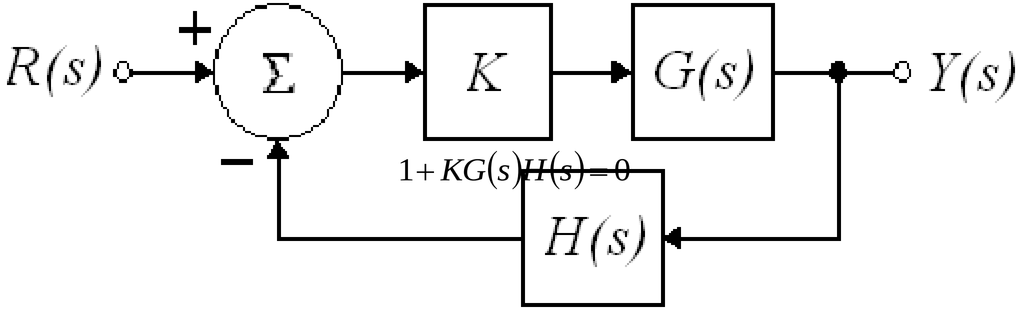
(usually the controller gain, *K*)

will modify the location of the closed-loop poles

in the *s*-domain.

Definition

- The closed-loop poles of the negative feedback control:



are the roots of the characteristic equation:

$$1+KG(s)H(s)=0$$

The root locus is the locus of the closed-loop poles when a specific parameter (usually gain, K) is varied from 0 to infinity.

Root Locus Method Foundations

- The value of *s* in the *s*-plane that make the loop gain *KG*(s)*H*(s) equal to -1 are the closed-loop poles

  (*i.e.* $1+KG(s)H(s)=0 \Leftrightarrow KG(s)H(s)=-1$)

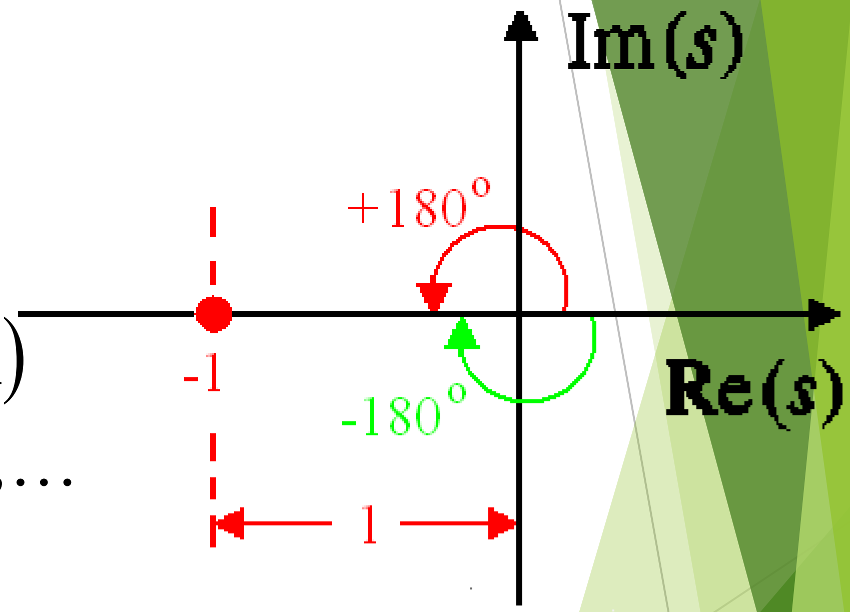
- *KG*(s)*H*(s) = -1 **can be split into two equations by equating the magnitudes and angles** of both sides of the equation.

Angle and Magnitude Conditions

$$KG(s)H(s) = -1$$

$$\Leftrightarrow \begin{cases} |KG(s)H(s)| = 1 \\ \angle KG(s)H(s) = \pm 180^0 (2l+1) \end{cases}$$

$$l = 0, 1, 2, \ldots$$



$$\Leftrightarrow \begin{cases} |G(s)H(s)| = 1/K \\ \angle G(s)H(s) = \pm 180^o (2l+1) \quad l = 0, 1, 2, \ldots \end{cases}$$

Independent of K

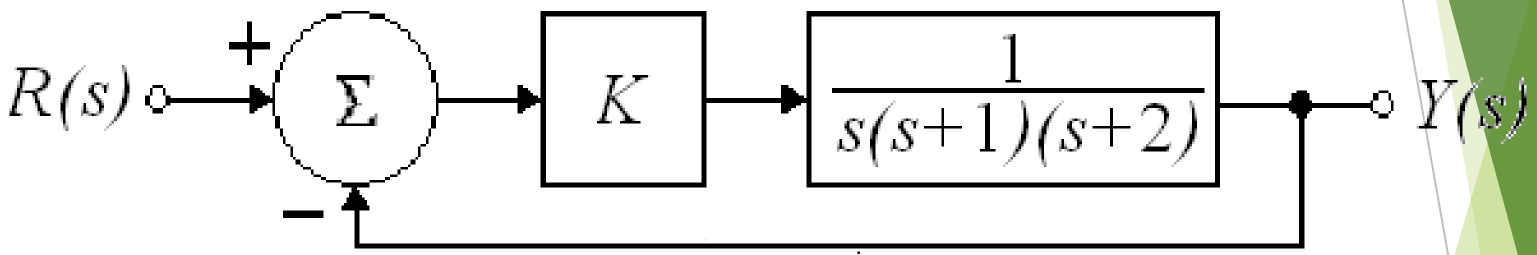
Root Locus Design Method

Part B: Drawing Root Locus *directly*

(= without solving for the closed-loop poles)

Learning by doing – Example 1

1) Sketch the root locus of the following system:



2) Determine the value of K such that the damping ratio ζ of a pair of dominant complex conjugate closed-loop is 0.5.

Rule #1

Assuming *n* poles and *m* zeros for *G*(*s*)*H*(*s*):

- The ***n*** **branches** of the root locus **start at the *n* poles.**
- ***m*** of these *n* branches **end on the *m* zeros**
- The ***n-m*** **other branches terminate at infinity** along asymptotes.

First step: Draw the *n* poles and *m* zeros of *G*(*s*)*H*(*s*) using x and o respectively

Applying Step #1

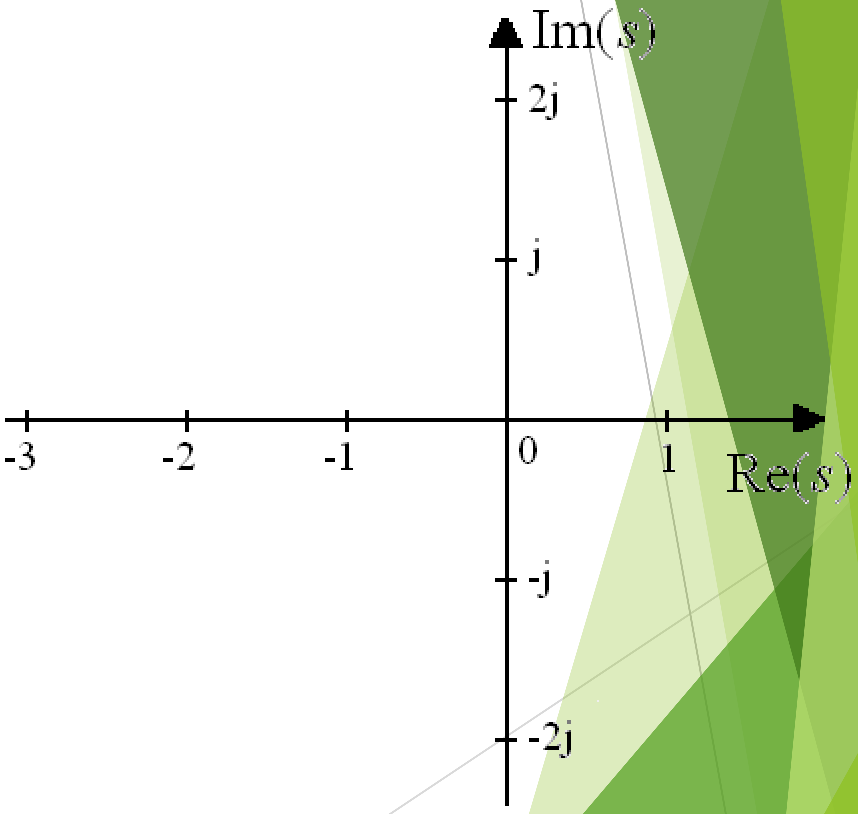
Draw the *n* poles and *m* zeros of $G(s)H(s)$ using x and o respectively.

$$G(s)H(s) = \frac{1}{s(s+1)(s+2)}$$

- 3 poles:

  $p_1 = 0; \quad p_2 = -1; \quad p_3 = -2$

- No zeros



Applying Step #1

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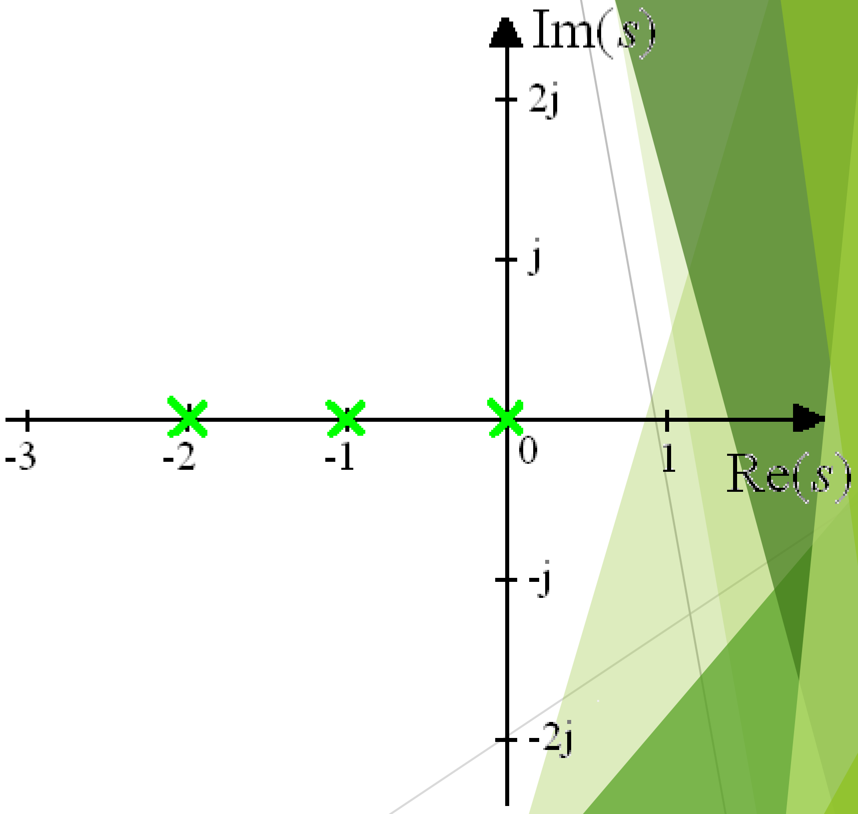


Rule #2

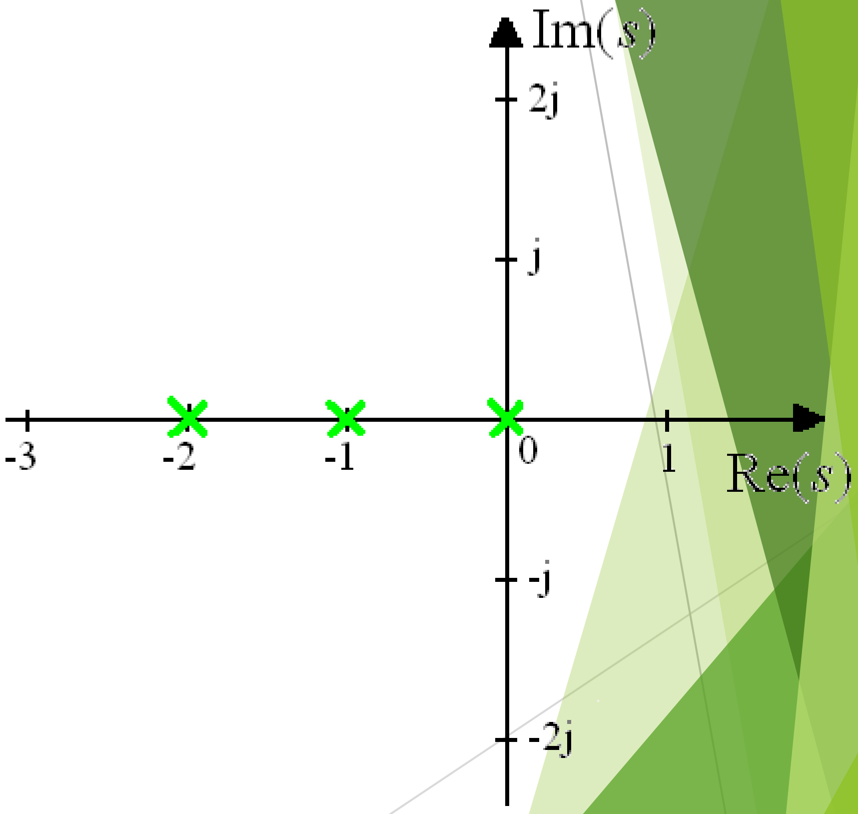
- The **loci on the real axis** are to the **left of an ODD number of REAL poles and REAL zeros** of $G(s)H(s)$

<u>Second step</u>: Determine the loci on the real axis. Choose a arbitrary test point. If the TOTAL number of both real poles and zeros is to the RIGHT of this point is ODD, then this point is on the root locus

Applying Step #2

Determine the loci on the real axis:

- Choose a arbitrary test point.
- If the TOTAL number of both real poles and zeros is to the RIGHT of this point is ODD, then this point is on the root locus



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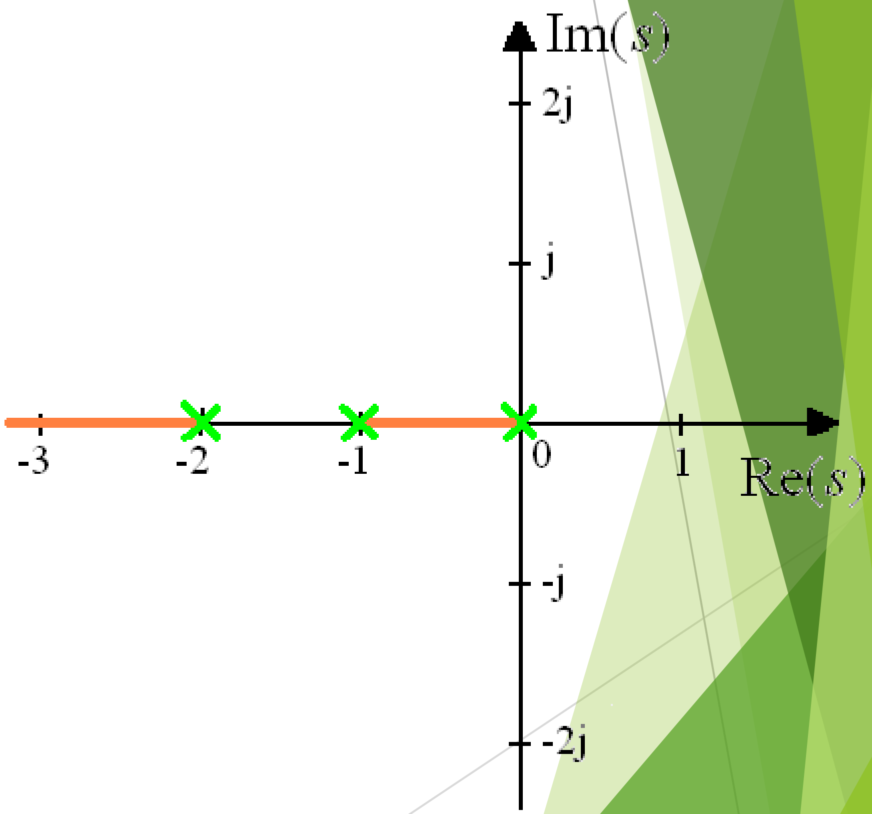


Rule #3

Assuming *n* poles and *m* zeros for *G*(*s*)*H*(*s*):

- The **root loci for very large values of s** must be **asymptotic** to **straight lines originate on the real axis at point:**

$$s = \alpha = \frac{\sum_{n} p_i - \sum_{m} z_i}{n - m}$$

radiating out from this point at angles:

$$\phi_l = \frac{\pm 180^o (2l + 1)}{n - m}$$

Third step: Determine the *n* - *m* asymptotes of the root loci. Locate s = α on the real axis. Compute and draw angles. Draw the asymptotes using dash lines.

Applying Step #3

Determine the *n - m* asymptotes:

- Locate s = α on the real axis:

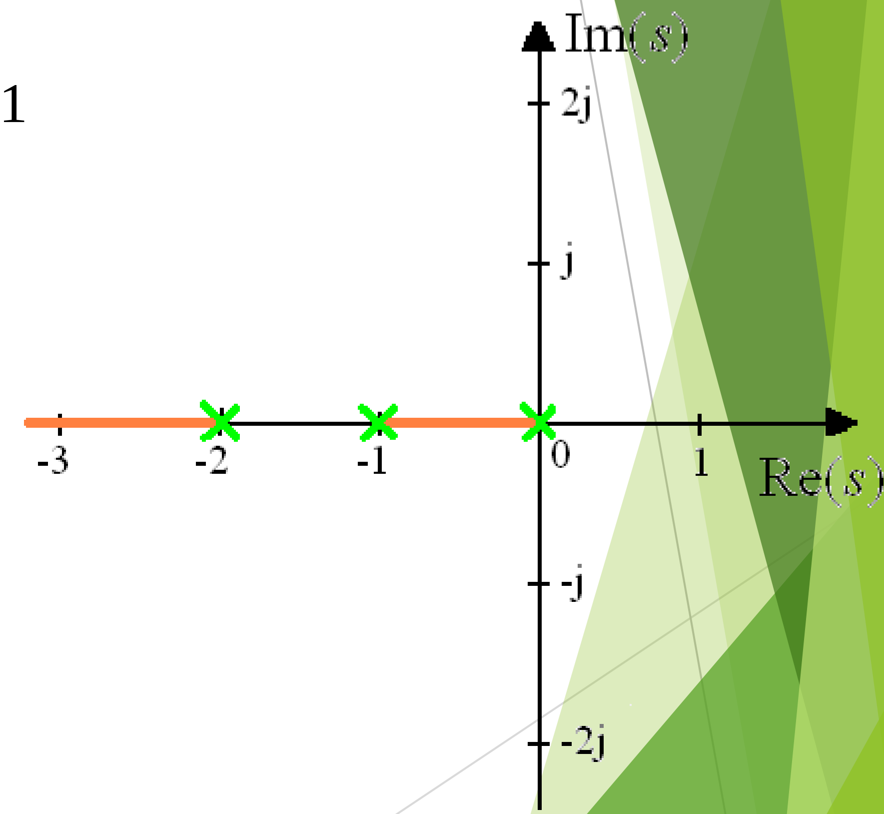
$$s=\alpha=\frac{p_1+p_2+p_3}{3-0}=\frac{0-1-2}{3}=-1$$

- Compute and draw angles:

$$\phi_l=\frac{\pm 180(2l+1)}{n-m}\qquad l=0,1,2,\ldots$$

$$\Rightarrow\begin{cases}\phi_0=\dfrac{\pm 180^0(2\times 0+1)}{3-0}=\pm 60^0\\[2ex]\phi_1=\dfrac{\pm 180^0(2\times 1+1)}{3-0}=\pm 180^0\end{cases}$$

- Draw the asymptotes using dash lines.



Applying Step #3

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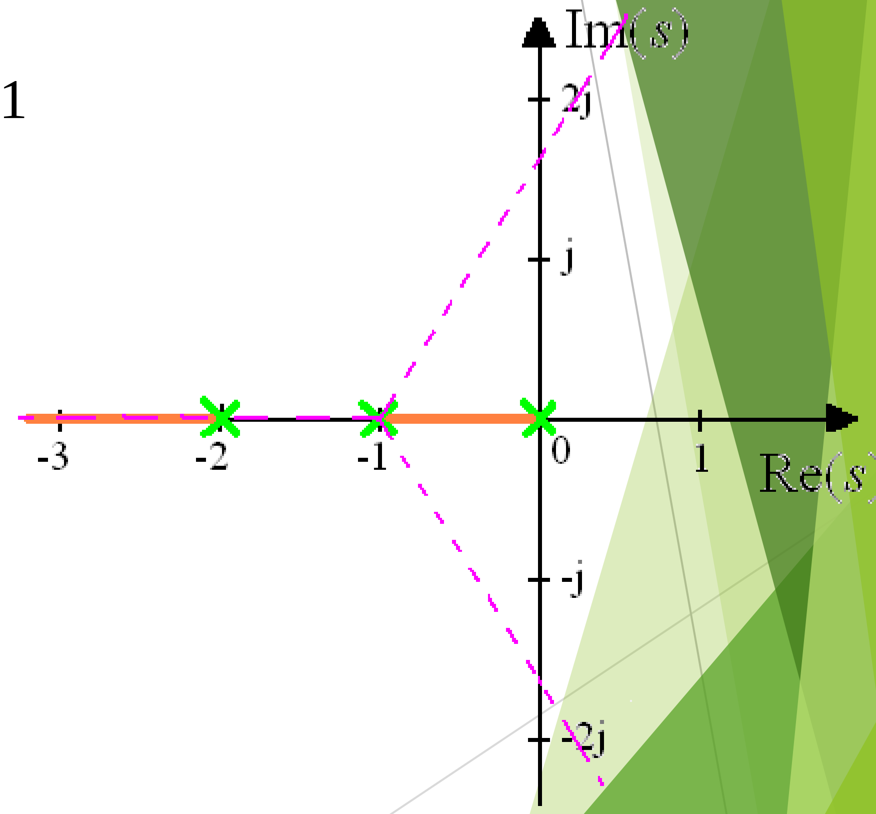


Breakpoint Definition

- The breakpoints are the points in the s-domain where **multiples** roots of the characteristic equation of the feedback control occur.

- These points correspond to intersection points on the root locus.

Rule #4

Given the characteristic equation is *KG*(s)*H*(s) = -1

- The **breakpoints are the closed-loop poles that satisfy:**

$$\frac{dK}{ds} = 0$$

Fourth step: Find the breakpoints. Express *K* such as:

$$K = \frac{-1}{G(s)H(s)}.$$

Set *dK*/*ds* = 0 and solve for the poles.

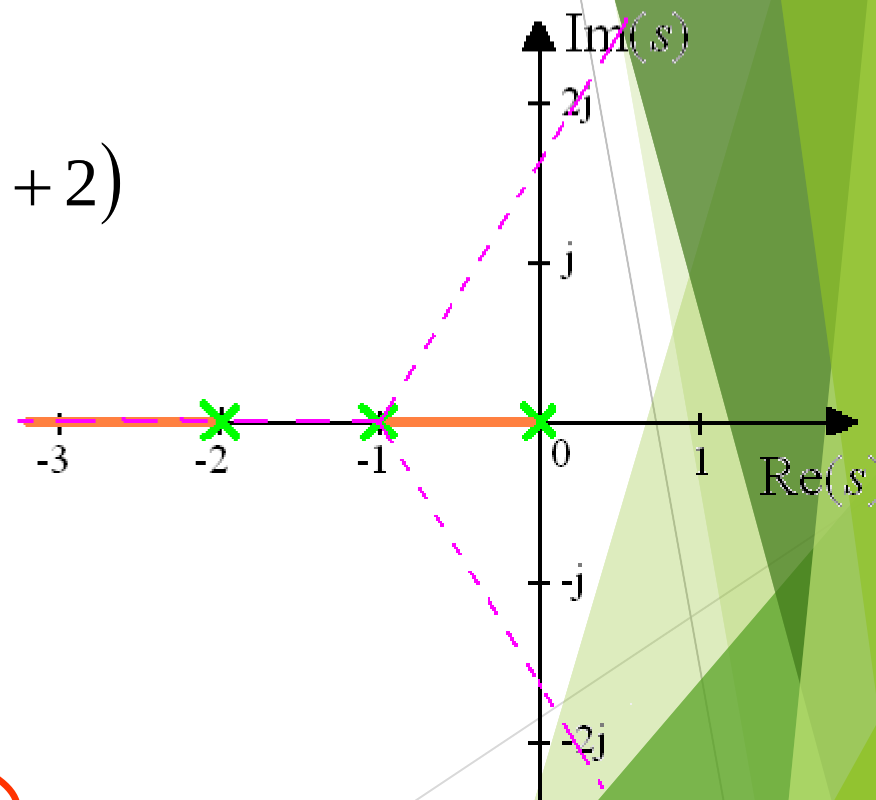
Applying Step #4

Find the breakpoints.

- Express *K* such as:

$$K = \frac{-1}{G(s)H(s)} = -s(s+1)(s+2)$$

$$K = -s^3 - 3s^2 - 2s$$



- Set *dK*/*ds* = 0 and solve for the poles.

$$-3s^2 - 6s - 2 = 0$$

$$s_1 = -1.5774, \quad s_2 = -0.4226$$

Applying Step #4

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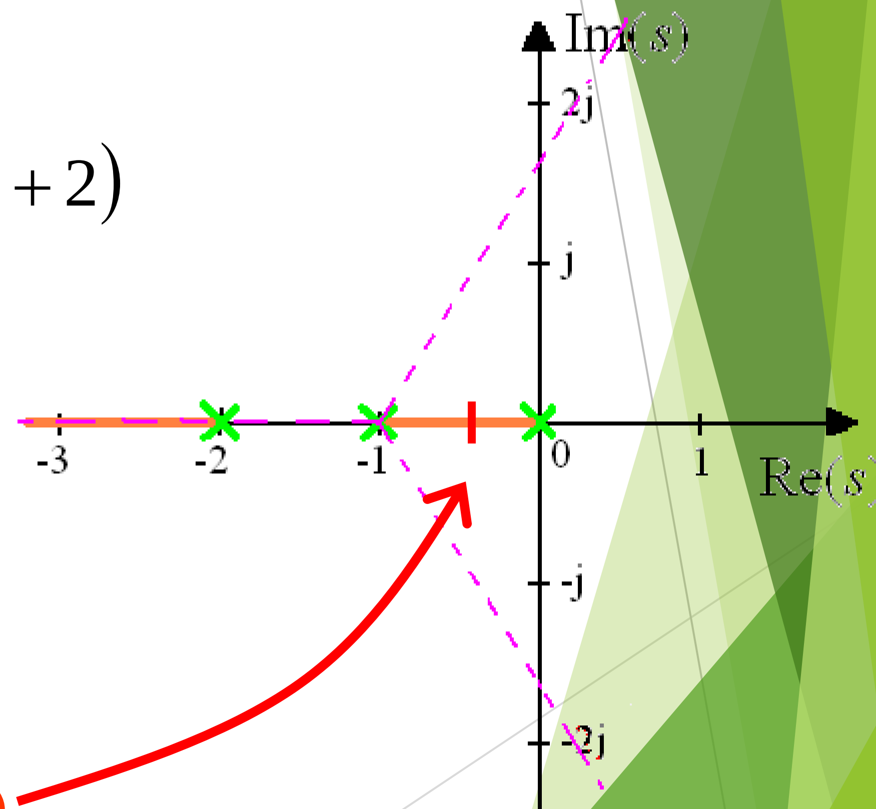


Recall Rule #1

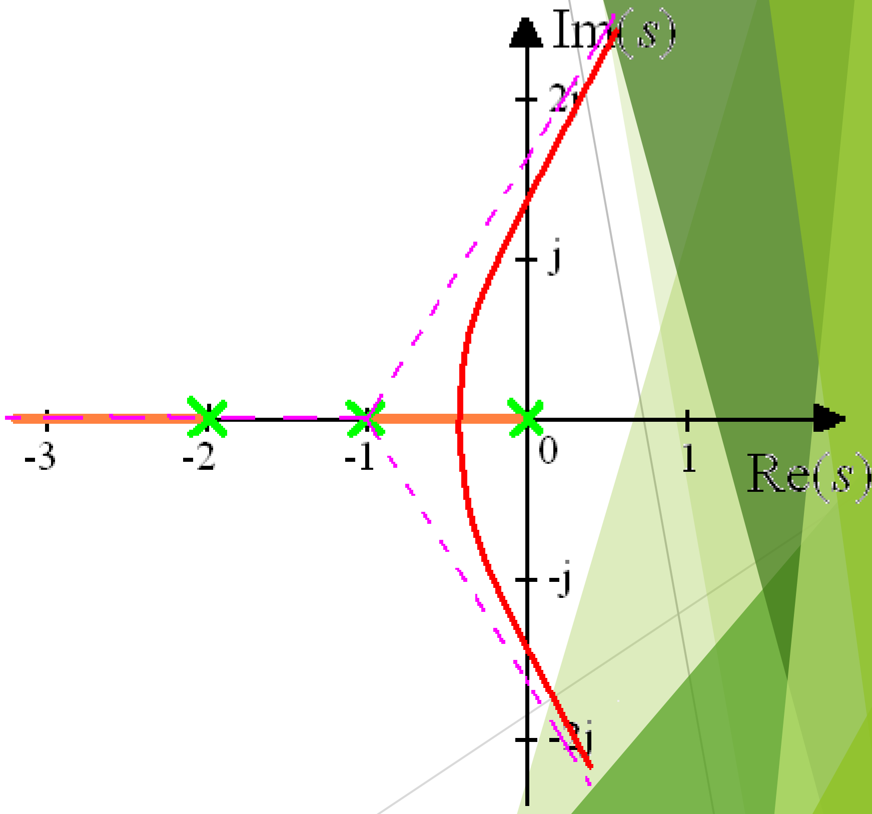
Assuming *n* poles and *m* zeros for *G*(*s*)*H*(*s*):

- The ***n*** **branches** of the root locus **start at the *n* poles.**
- ***m*** of these *n* branches **end on the *m* zeros**
- The ***n-m*** **other branches terminate at infinity** along asymptotes.

Last step: Draw the *n*-*m* branches that terminate at infinity along asymptotes

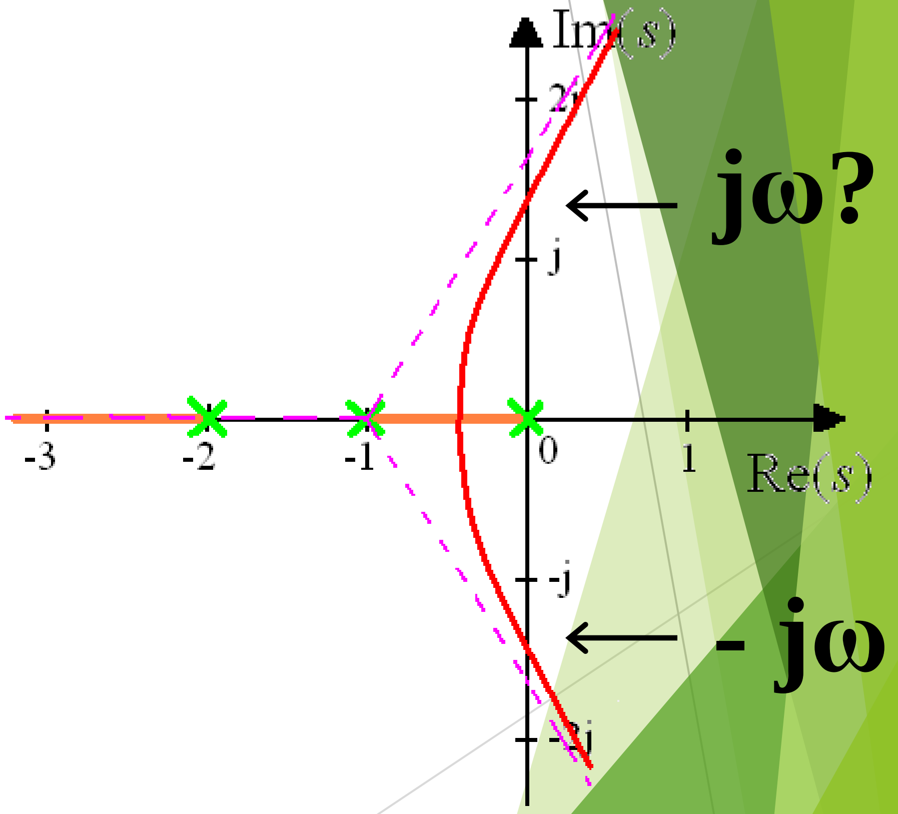
Applying Last Step

- Draw the *n-m* branches that terminate at infinity along asymptotes



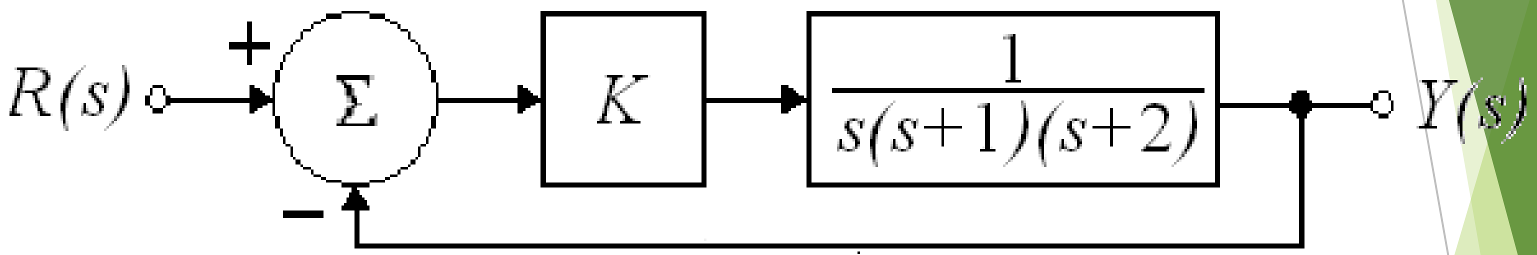
Points on both root locus & imaginary axis?

- Points on imaginary axis satisfy: $$s = j\omega$$
- Points on root locus satisfy: $$1 + KG(s)H(s) = 0$$
- Substitute $s = j\omega$ into the characteristic equation and solve for ω. $$\omega = 0 \quad or \quad \omega = \pm\sqrt{2}$$



Learning by doing – Example 1

1) Sketch the root locus of the following system:

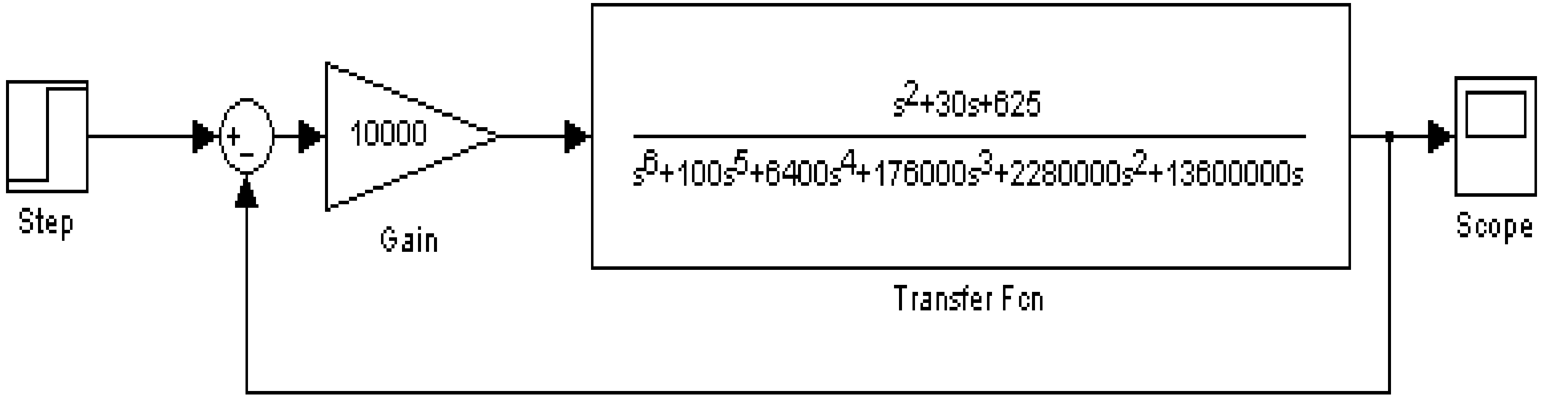


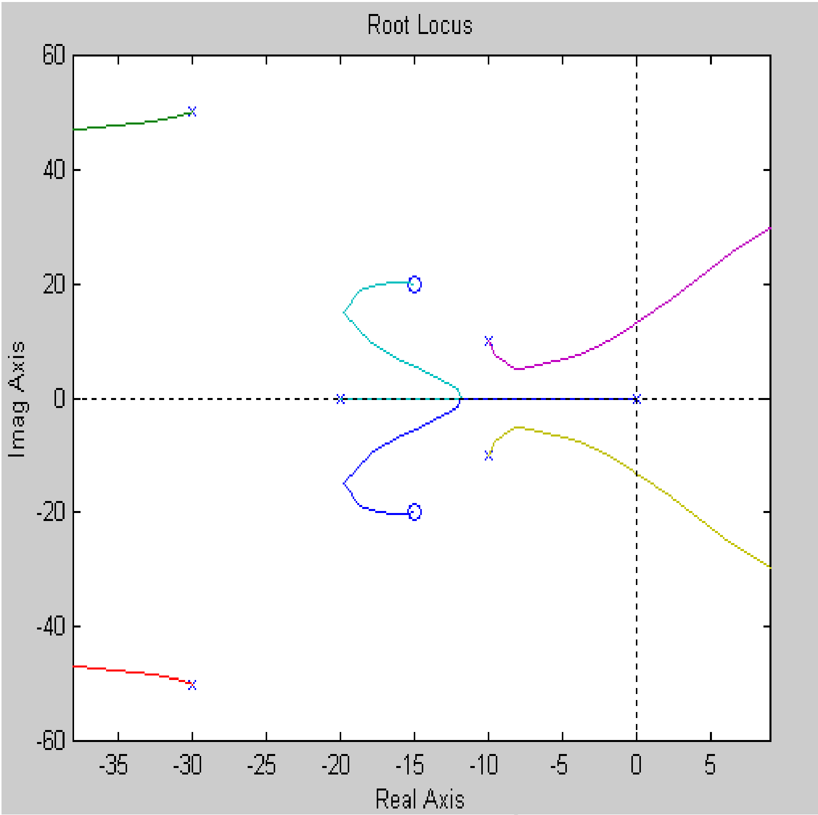
2) Determine the value of K such that the damping ratio ζ of a pair of dominant complex conjugate closed-loop is 0.5.

→

See class notes

The Root Locus Method



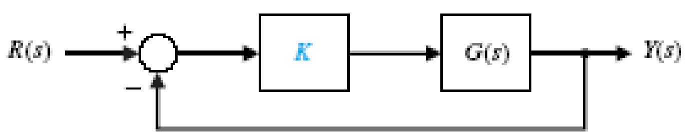


The Root   Locus Method

The root locus is a graphical procedure for determining the poles of a closed-loop system given the poles and zeros of a forward-loop system. Graphically, the locus is the set of paths in the complex plane traced by the closed-loop poles as the root locus gain is varied from zero to infinity.

In mathematical terms, given a forward-loop transfer function, K**G**(s) where K is the root locus gain, and the corresponding closed-loop transfer function

$$\frac{KG(s)}{1+KG(s)}$$

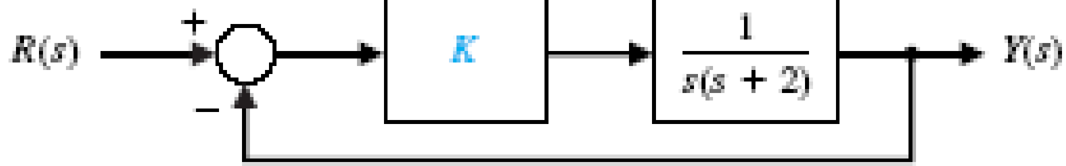


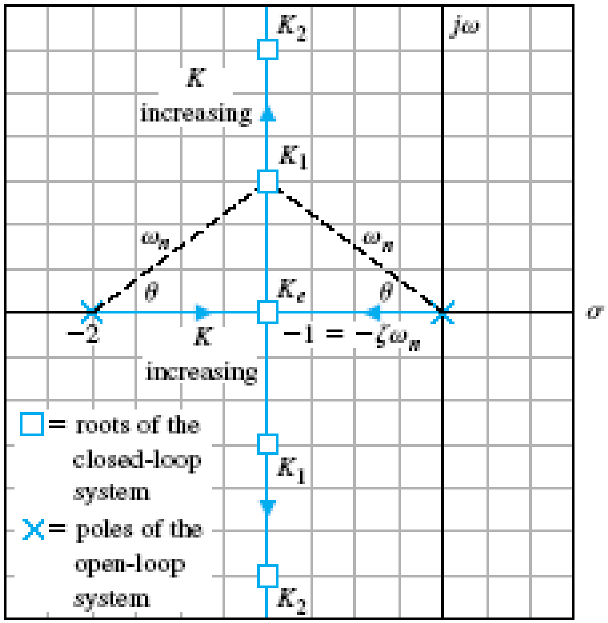
the root locus is the set of paths traced by the roots of

$$1 + KG(s) = 0$$

as K varies from zero to infinity. As K changes, the solution to this equation changes. This equation is called the characteristic equation. This equation defines where the poles will be located for any value of the root locus gain, K. In other words, it defines the characteristics of the system behavior for various values of controller gain.

The Root Locus Method





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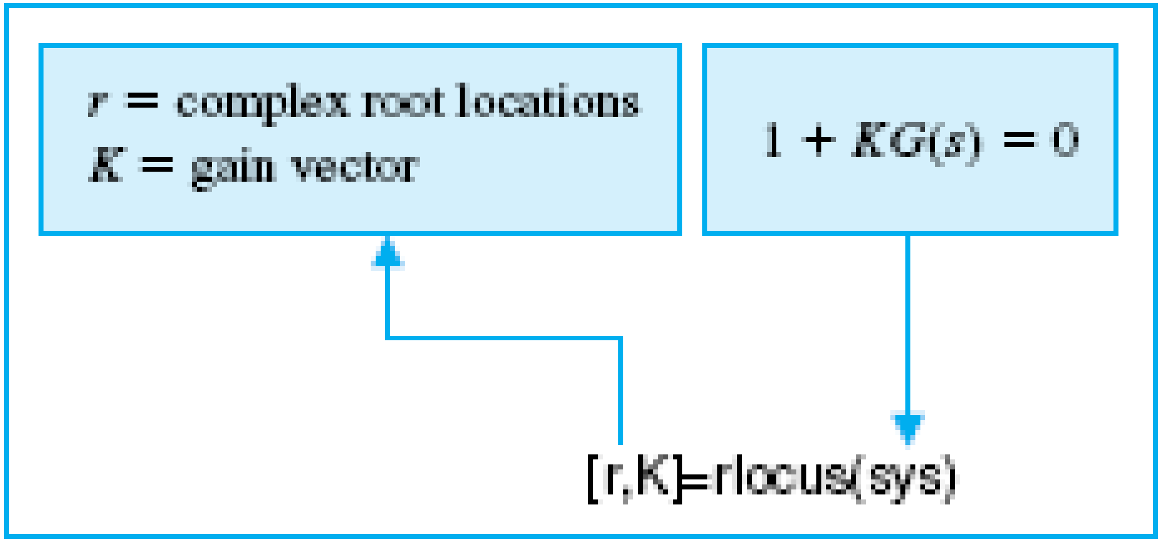

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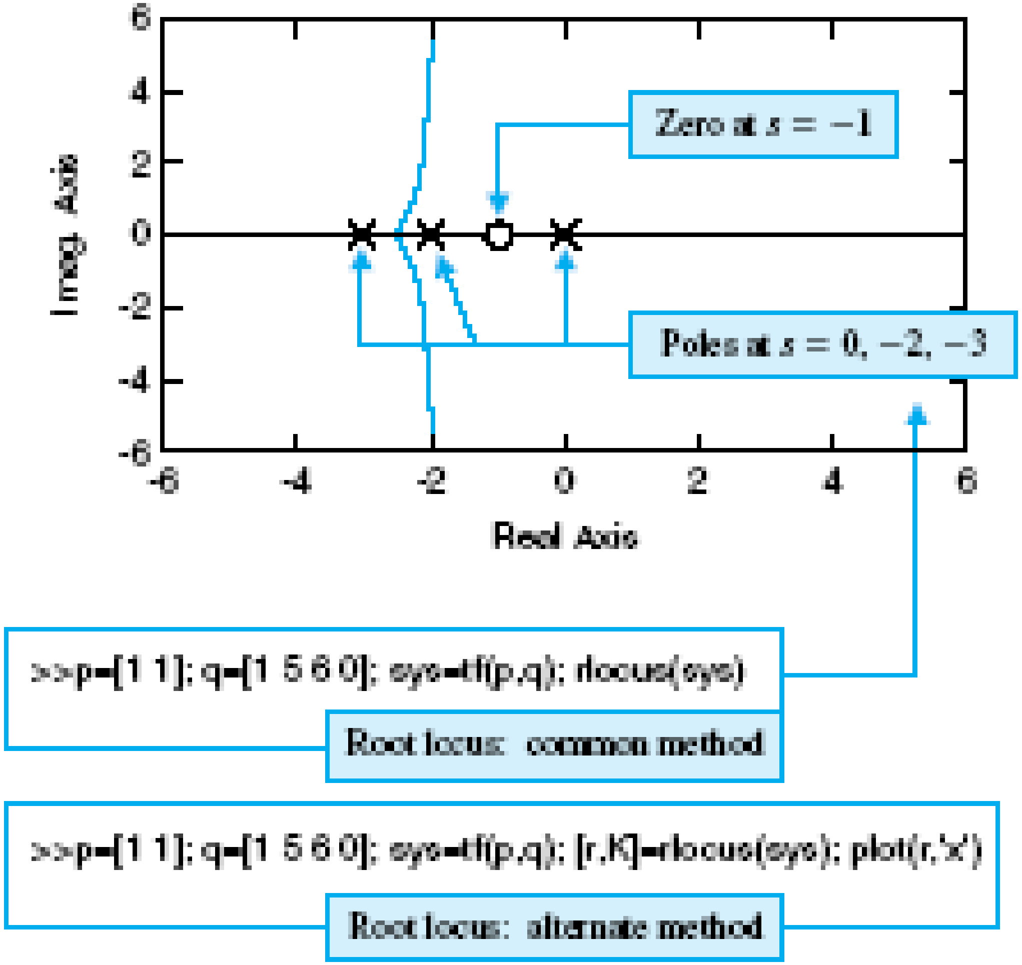

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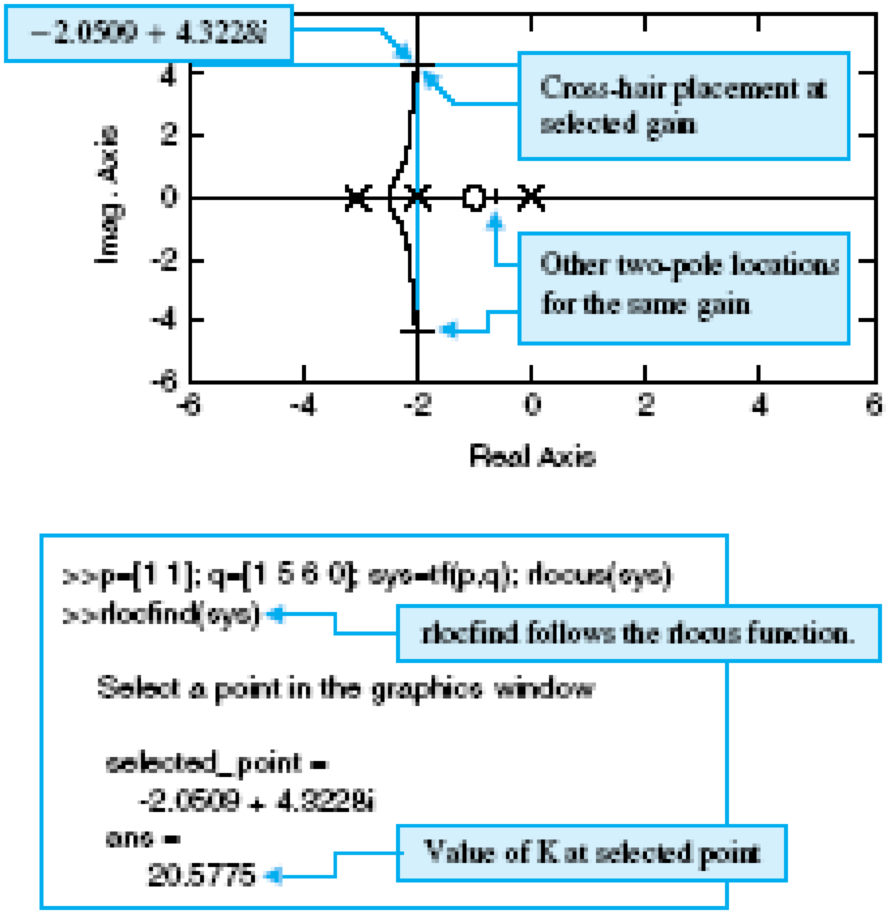


The Root Locus Method

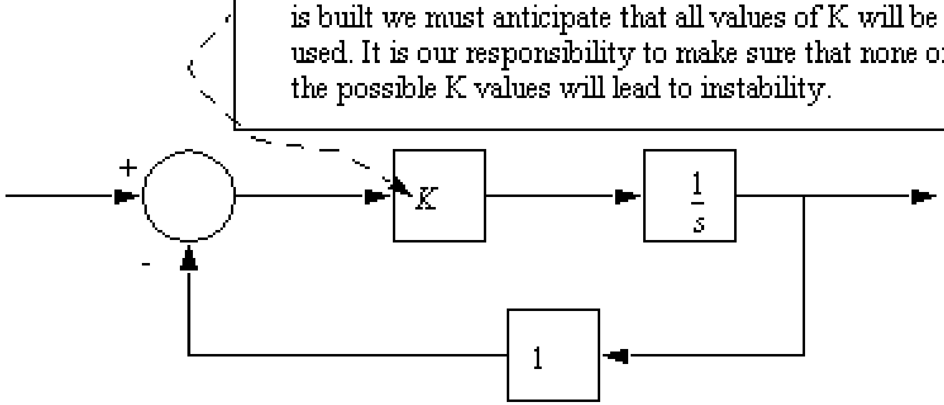
No matter what we pick K to be, the closed-loop system must always have n poles, where n is the number of poles of G(s).
The root locus must have n branches, each branch starts at a pole of G(s) and goes to a zero of G(s).
If G(s) has more poles than zeros (as is often the case), m < n and we say that G(s) has zeros at infinity. In this case, the limit of G(s) as s -> infinity is zero.

The number of zeros at infinity is n-m, the number of poles minus the number of zeros, and is the number of branches of the root locus that go to infinity (asymptotes).

Since the root locus is actually the locations of all possible closed loop poles, from the root locus we can select a gain such that our closed-loop system will perform the way we want. If any of the selected poles are on the right half plane, the closed-loop system will be unstable. The poles that are closest to the imaginary axis have the greatest influence on the closed-loop response, so even though the system has three or four poles, it may still act like a second or even first order system depending on the location(s) of the dominant pole(s).

Example

Note: This controller has adjustable gain. After this design is built we must anticipate that all values of K will be used. It is our responsibility to make sure that none of the possible K values will lead to instability.



$$G(s) = \frac{K}{s} \qquad H(s) = 1$$

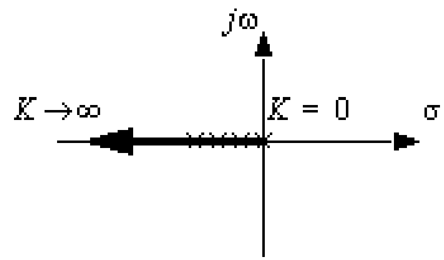
First, we must develop a transfer function for the entire control system.

$$G_S(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{\left(\frac{K}{s}\right)}{1 + \left(\frac{K}{s}\right)(1)} = \frac{K}{s + K}$$

Next, we use the characteristic equation of the denominator to find the roots as the value of K varies. These can then be plotted on a complex plane. Note: the value of gain 'K' is normally found from 0 to +infinity.

$$s + K = 0$$

K	root
0	
1	
2	
3	
etc..	



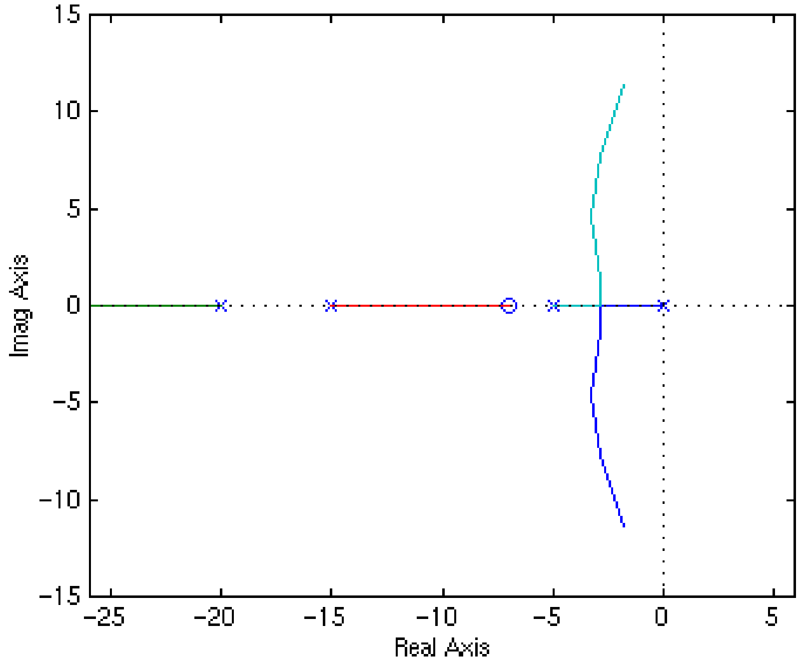
Note: because all of the roots for all values of K are real negative this system will always be stable, and it will always tend to have a damped response. The large the value of K, the more stable the system becomes.

Example

MATLAB Example - Plotting the root locus of a transfer function

Consider an open loop system which has a transfer function of
G(s) = (s+7)/s(s+5)(s+15)(s+20)
How do we design a feedback controller for the system by using the root locus method?
Enter the transfer function, and the command to plot the root locus:

```
num=[1 7];
den=conv(conv([1 0],[1 5]),conv([1 15],[1 20]));
rlocus(num,den)
axis([-22 3 -15 15])
```



Graphical Method

Given the system elements (you should assume negative feedback),

$$G(s) = \frac{K}{s^2+3s+2} \qquad H(s) = 1$$

Step 1: (put equation in standard form)

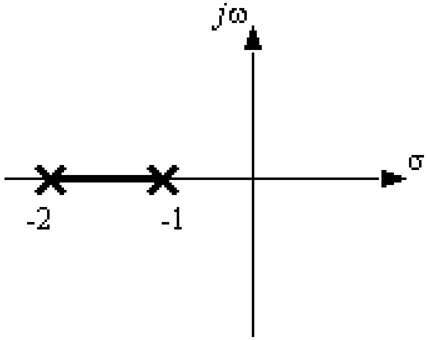
$$1 + G(s)H(s) = 1 + \left(\frac{K}{s^2+3s+2}\right)(1) = 1 + K\frac{1}{(s+1)(s+2)}$$

Step 2: (find loci ending at infinity)

$m = 0 \quad n = 2$ (from the poles and zeros of the previous step)

$n - m = 2$ (loci end at infinity)

Step 3: (plot roots)



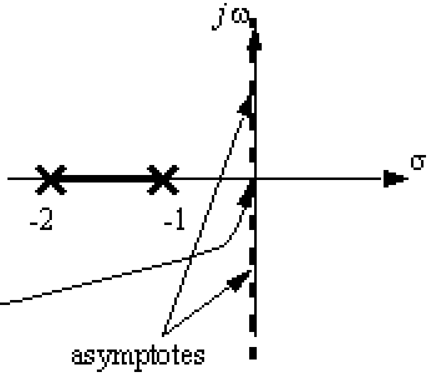
Step 4: (find asymptotes angles and real axis intersection)

$$\beta(k) = \frac{180°(2k+1)}{2} \qquad k \in I[0,1]$$

$$\beta(0) = \frac{180°(2(0)+1)}{2} = 90°$$

$$\beta(1) = \frac{180°(2(1)+1)}{2} = 270°$$

$$\sigma = \frac{(0)(-1-2)}{2} = 0$$



Graphical Method

Step 5: (find the breakout points for the roots)

$$A = 1 \qquad B = s^2 + 3s + 2$$

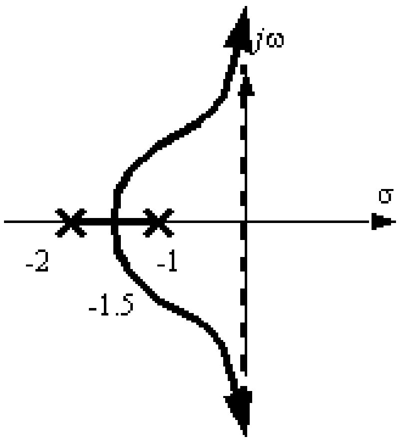
$$\frac{d}{ds}A = 0 \qquad \frac{d}{ds}B = 2s + 3$$

$$A\left(\frac{d}{ds}B\right) - B\left(\frac{d}{ds}A\right) = 0$$

$$1(2s+3) - (s^2+3s+2)(0) = 0$$

$$2s + 3 = 0$$

$$s = -1.5$$



Note: because the loci do not intersect the imaginary axis, we know the system will be stable, so step 6 is not necessary, but we it will be done for illustrative purposes.

Step 6: (find the imaginary intercepts)

$$1 + G(s)H(s) = 0$$

$$1 + K\frac{1}{s^2+3s+2} = 0$$

$$s^2 + 3s + 2 + K = 0$$

$$(j\omega)^2 + 3(j\omega) + 2 + K = 0$$

$$-\omega^2 + 3j\omega + 2 + K = 0$$

$$\omega^2 + \omega(-3j) + (-2 - K) = 0$$

$$\omega = \frac{3j \pm \sqrt{(-3j)^2 - 4(-2-K)}}{2} = \frac{3j \pm \sqrt{-9+8+4K}}{2} = \frac{3j \pm \sqrt{4K-1}}{2}$$

In this case the frequency has an imaginary value. This means that there will be no frequency that will intercept the imaginary axis.

Root Locus Design GUI (rltool)

The Root Locus Design GUI is an interactive graphical tool to design compensators using the root locus method. This GUI plots the locus of the closed-loop poles as a function of the compensator gains. You can use this GUI to add compensator poles and zeros and analyze how their location affects the root locus and various time and frequency domain responses. Click on the various controls on the GUI to see what they do.